

Ion Acoustic Turbulence and Anomalous Transport

W. Horton¹

Theoretical interpretation of ion acoustic turbulence is shown to require the use of renormalized turbulence theory for calculating the turbulent spectra and transport coefficients. The physics of solitons, double layers, and ion phase space holes have an impact on the one-dimensional problem.

KEY WORDS: Anomalous transport; turbulence; anomalous resistivity; ion acoustic waves; ion acoustic solitons.

1. ION ACOUSTIC PROBLEM

Ion acoustic turbulence is driven in a collisionless plasma by a wide range of mechanisms including the injection of laser beams or particle beams and through the passage of high currents. In this review we consider principally the case where the ion acoustic waves given by $\omega(\mathbf{k}) = kc_s / (1 + k^2 \lambda_{De}^2)^{1/2}$ with the ion sound speed $c_s = (T_e/m_i)^{1/2}$ and the Debye length $\lambda_{De} = (T_e/4\pi ne^2)^{1/2}$ are driven unstable by a current $\mathbf{j} = -env_d$. The drift of the thermal electrons through the ions with velocity \mathbf{v}_d produces linear instability with $\gamma^e(\mathbf{k}) = (\pi/8)^{1/2} (m_e/m_i)^{1/2} (\mathbf{k} \cdot \mathbf{v}_d - \omega_{\mathbf{k}})$ from the positive slope on the electron velocity distribution for $v_d > \omega_{\mathbf{k}}/k = c_s$. For $T_e \gg T_i$ and $v_d \gg c_s$ the ion Landau damping $\gamma^i(\mathbf{k}) = -(\pi/8)^{1/2} (T_e/T_i)^{3/2} \omega_{\mathbf{k}} \exp(-T_e/2T_i)$ is negligible and all \mathbf{k} modes within the polar angle $\vartheta \leq \vartheta_0$ with respect to \mathbf{v}_d where $\cos \vartheta_0 = c_s/v_d$ are linearly unstable.

Examples of current-driven ion acoustic turbulence occur in shocks,⁽¹⁾ magnetic reconnection,⁽²⁻⁵⁾ plasma return currents⁽⁶⁾ arising from the current neutralization of injected particle beams and turbulent heating experiments.^(7,8)

¹ University of Texas, Austin, Texas 78712.

The principal effect of ion acoustic turbulence⁽⁶⁻¹¹⁾ is to produce an anomalous resistivity $\eta = m_e v_{\text{eff}}/ne^2$ and the associated turbulent heating^(7,8) through the scattering of the electrons from the ion acoustic fluctuations. Numerous experimental studies of turbulent heating show the presence of ion acoustic turbulence from the measurements of the fluctuation spectra. The principal characteristics of the system are the presence of a turbulent heating pulse with $\omega_{pi}At \sim 100$ during which a strong resistivity and dc electric field $\langle E \rangle$ arise with $e\langle E \rangle = v_{\text{eff}}m_e v_d$. The effective collision frequency $v_{\text{eff}} \lesssim \omega_{pi}$ typically exceeds the collisional resistivity by several orders of magnitude. The turbulent heating pulse produces a conversion of drift electron kinetic energy $\frac{1}{2}n_e m_e v_d^2$ into thermal energy $n_e T_e$ through $j \cdot E \Delta t = n_e m_e v_d^2 v_{\text{eff}} \Delta t$. The momentum of the drifting electrons $n_e m_e v_d$ is taken up by the ion acoustic waves propagating in the direction of v_d and then transferred to a small fraction of fast ions⁽¹²⁻¹⁴⁾ $n_f/n_e \sim (m_e/m_i)^{1/4}$ with $T_f \sim \frac{1}{3}T_e$. The production of the fast ion population arises from the turbulent trapping of the ions by the finite amplitude ion acoustic waves.⁽¹⁵⁾

The detailed interpretation of a turbulent heating experiment requires that account be taken of the losses of the electron thermal energy, of the fast ion component, and of the waves from the region of turbulent heating. The turbulent heating and loss processes taken together then determine the long-time macroscopic evolution of the system.⁽¹⁶⁾ Current penetration and net thermal deposition or efficiency are determined from the macroscopic balance analysis. Here we are only concerned with the microscopic laws for the anomalous resistivity arising from ion acoustic turbulence. The microscopic laws are the basis for the macroscopic confinement and heating studies.^(7,16)

2. COLLECTIVE INTERACTIONS BETWEEN ELECTRONS AND IONS

The ion acoustic waves and the associated solitons and double layers, are the mechanism for the interactions between electrons and ions in a collisionless plasma. During the slow ion acoustic motion the electrons remain near equilibrium with $-en_e \mathbf{E}(\mathbf{x}, t) = \nabla(n_e T_e) \approx T_e \nabla n_e$ and the ions are accelerated by the electric field according to $\partial v + v \cdot \nabla v = (e_i/m_i) \mathbf{E}(\mathbf{x}, t)$. The ion density changes by $\partial_t n_i(\mathbf{x}, t) = -\nabla \cdot (n_i v)$. The self-consistent long-range Coulomb electric field is determined by

$$\nabla \cdot \mathbf{E} = -\nabla^2 \varphi(\mathbf{x}, t) = 4\pi e(n_i - n_e) \quad (1)$$

For small-amplitude oscillations these closed fluid equations yield $\omega_{\mathbf{k}} = kc_s/(1 + k^2 \lambda_{De}^2)^{1/2}$ and for wavelengths long compared with the Debye

length $\omega_k = |\mathbf{k}| c_s$. For large amplitudes the steepening nonlinearity $v \partial_x v$ of the fluid equations balances with the dispersion from $\omega_k = kc_s(1 - \frac{1}{2}k^2 \lambda_{De}^2)$ to produce ion acoustic solitons propagating with speeds $c > c_s$. We discuss the soliton components in Section 5. To describe the interactions of the particles with waves requires the Vlasov equations.

For finite amplitude potential fluctuations $\varphi_{\mathbf{k}\omega} \exp(i\mathbf{k} \cdot \mathbf{x} - i\omega t)$ the fluctuation $\delta f_{\mathbf{k}\omega}(\mathbf{v}) \exp(i\mathbf{k} \cdot \mathbf{x} - i\omega t)$ of the particle distribution function satisfies

$$(\omega - \mathbf{k} \cdot \mathbf{v}) \delta f_{\mathbf{k}\omega}(\mathbf{v}) = -\frac{e}{m} \varphi_k \mathbf{k} \cdot \frac{\partial \langle f \rangle}{\partial \mathbf{v}} - \frac{e}{m} \sum_{k_1 \neq k} \varphi_{k_1} \mathbf{k}_1 \cdot \frac{\partial}{\partial \mathbf{v}} \delta f_{k-k_1} \quad (2)$$

and the mean distribution evolves by

$$\frac{\partial \langle f \rangle}{\partial t} = -\frac{e}{m} \frac{\partial}{\partial \mathbf{v}} \cdot \left\langle \sum_{\mathbf{k}\omega} i\mathbf{k} \varphi_{\mathbf{k}\omega}^* \delta f_{\mathbf{k}\omega}(\mathbf{v}) \right\rangle \quad (3)$$

where $k = |\mathbf{k}\omega$. In the limit $\varphi_k \rightarrow 0$ we write from Eq. (2)

$$\delta f_{\mathbf{k}\omega}(\mathbf{v}) = g_{\mathbf{k}\omega}^0(v) \left(\frac{-e}{m} \varphi_k \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{v}} \right) \langle f \rangle \equiv \frac{k}{k} \langle f \rangle$$

and obtain the self-consistent linear modes $\varepsilon_k^1(\omega) \varphi_k = 0$ from Eq. (1) with

$$\varepsilon_k^1(\omega) = 1 + \frac{\omega_p^2}{k^2} \int d\mathbf{v} g_{\mathbf{k}\omega}^0 \mathbf{k} \cdot \frac{\partial \langle f \rangle}{\partial \mathbf{v}} \cong 1 + \frac{\omega_{pe}^2}{k^2 \lambda_{De}^2} - \frac{\omega_{pi}^2}{\omega^2} + i\varepsilon_k'' \quad (4)$$

with

$$\varepsilon_k'' = -\frac{\omega_p^2}{k^2} \int d\mathbf{v} \pi \delta(\omega - \mathbf{k} \cdot \mathbf{v}) \mathbf{k} \cdot \frac{\partial \langle f \rangle}{\partial \mathbf{v}}$$

where $g_{\mathbf{k}\omega}^0(\mathbf{v}) = (\omega - \mathbf{k} \cdot \mathbf{v} + i0^+)^{-1}$ is the linear particle propagator and ε_k'' gives the response of the resonant particles which we now consider in more detail.

A nearly resonant electron $\omega \cong \mathbf{k} \cdot \mathbf{v}$ is accelerated in one wave period $1/\omega_k$ by

$$\Delta V = \frac{e\mathbf{k}\varphi_k}{m_e(\omega - \mathbf{k} \cdot \mathbf{v}_e)} = \frac{e\mathbf{k}\varphi_k}{m_e \omega_k}$$

The electron is accelerated out of resonance when $\Delta V > \omega/k$, which requires

$$\Delta V > \frac{\omega}{k} \Rightarrow \frac{e\varphi_k}{T_e} > \left(\frac{\omega_k}{kv_e} \right)^2 \simeq \frac{m_e}{m_i} \quad (5)$$

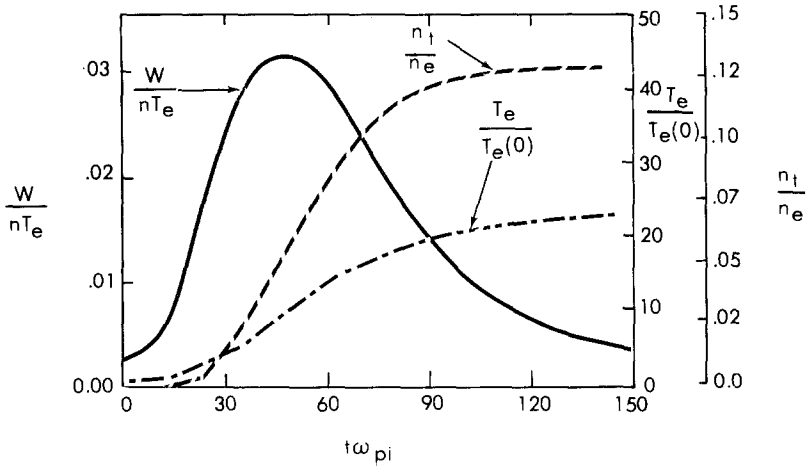


Fig. 1. Turbulent heating pulse computed from mode simulation with renormalized turbulence theory using $v_d = v_e(0)$, $m_i/m_e = 1600$, and $T_e(0)/T_i(0) = 50$.

The acceleration out of resonance is not described by the linear propagator $g_{\mathbf{k}\omega}^0(\mathbf{v})$. The expansion of $\Delta V(\mathbf{k}, \mathbf{v})$ in powers of $g_{\mathbf{k}\omega}^0(\mathbf{v})$ fails for amplitudes greater than the critical amplitude given in Eq. (5). In fact, we know for a single resonant particle the nonlinear motion can be expressed in terms of the pendulum equation $k\ddot{x} = -\omega_b^2 \sin kx$ with $\omega_b^2 = ek^2\varphi_k/m_e$. For two waves the motion becomes stochastic at the very low amplitude $v_i(k) = (e\varphi_k/m_e)^{1/2} \lesssim (\omega/k) = c_s$. Study of the test particle with the Hamiltonian $H = \frac{1}{2}m_e v^2 - e\varphi(\mathbf{x}, t)$ gives the conditions for stochastic motion for given wave spectra.

The condition for stochastic motion and the form of the velocity space diffusion $D(\mathbf{v})$ depends on the dimensionality d of the system. The resonant domains satisfy $\omega = \mathbf{k} \cdot \mathbf{v}$ which is a $(d-1)$ -dimensional velocity subspace. For $d=1$ there is a unique resonant velocity, for $d=2$ the resonant velocities occur on a line and for $d=3$ the resonant velocities lie on a surface. For this reason the $d=1$ system evolves qualitatively differently, forming a quasilinear plateau, whereas the $d=2$ (simulations) and $d=3$ (laboratory experiments) do not form a plateau but evolve through a turbulent heating pulse as shown in Fig. 1.

3. LIMITATIONS OF WEAK TURBULENCE THEORY

It is the assumption of weak turbulence theory⁽¹¹⁾ that the dispersion in $\Delta \mathbf{k}$ and $\Delta \mathbf{v}$ in the linear resonance $\omega = \mathbf{k} \cdot \mathbf{v}$ is sufficient to justify the use of the small amplitude expansion in powers of $g_{\mathbf{k}}^0(\mathbf{v}) = (\omega - \mathbf{k} \cdot \mathbf{v} + i0^+)^{-1}$.

For ion acoustic turbulence the dispersion is known to increase the range of validity of the expansion in $g_k^0(\mathbf{v})$ well above the single particle limit (3), but not to the amplitude levels of primary interest for ion acoustic turbulence.

To concisely discuss the weak turbulence theory and its limits of validity we introduce the diagrammatic notation of Refs. 17–20. The third-order perturbation expansion used in weak turbulence theory is

$$\delta f_k^{\text{WT}}(\mathbf{v}) = \downarrow_k + \begin{array}{c} |k_1 \\ \downarrow \\ k \quad k-k_1 \end{array} + \begin{array}{c} |k_1 \quad |k_2 \\ \downarrow \\ k \quad k-k_1 \quad k-k_1-k_2 \end{array} \quad (6)$$

where $g_k^0(\mathbf{v}) = \bar{k}$ and $\sum_{k_1} (-e/m) \varphi_{k_1} \mathbf{k}_1 \cdot \partial_{\mathbf{v}} = |k_1$ with the summation \sum_{k_1} deleted for \downarrow^{k_1} . Terms of order $|(e/m) g_k^0 \varphi_k \mathbf{k} \cdot \partial_{\mathbf{v}}|^4$ are dropped. The charge density $\rho_k^{\text{WT}} = \sum e \int \delta f_k^{\text{WT}} d\mathbf{v}$ is third order and yields the nonlinear Poisson field equation

$$\varepsilon_k^1 \varphi_k + \sum_{\substack{k_1+k_2 \\ =k}} \varepsilon_{k_1, k_2}^{(2)} \varphi_{k_1} \varphi_{k_2} + \sum_{\substack{k_1+k_2+k_3 \\ =k}} \varepsilon_{k_1, k_2, k_3}^{(3)} \varphi_{k_1} \varphi_{k_2} \varphi_{k_3} = 0 \quad (7)$$

determining φ_k . Here ε_k^1 is given in Eq. (4) and $\varepsilon_{1,2}^{(2)}$ and $\varepsilon_{1,2,3}^{(3)}$ are the nonlinear dielectrics given in Eqs. (2.21) and (2.22) of Ref. 19.

The approximate radius of convergence of expansion (7) is determined by requiring that the third-order contribution in (7) be less than the first order.

To determine the contributions of $\varepsilon_k^{(2),(3)}$ to the dynamics the assumption is made that because of the $A\mathbf{k}$ dispersion the coupling between the modes is weak. The long time scale $\omega_k T \gg 1$ evolution of the spectrum $I_{\mathbf{k}}(T) = \langle |\varphi_{\mathbf{k}}|^2 \rangle$ is determined by calculating $\langle \varphi_1 \varphi_2 \varphi_3 \rangle$ perturbatively in terms of $\langle \varphi_1 \varphi_2 \varphi_3 \varphi_4 \rangle$ and closing with the Gaussian or quasnormal approximation

$$\langle \varphi_1 \varphi_2 \varphi_3 \varphi_4 \rangle = \langle \varphi_1 \varphi_2 \rangle \langle \varphi_3 \varphi_4 \rangle + \langle \varphi_1 \varphi_3 \rangle \langle \varphi_2 \varphi_4 \rangle + \langle \varphi_1 \varphi_4 \rangle \langle \varphi_2 \varphi_3 \rangle \quad (8)$$

The evolution of $I_{\mathbf{k}}(T)$ is given by

$$\begin{aligned} \frac{\partial \varepsilon_k'}{\partial \omega_k} \frac{dI_{\mathbf{k}}}{dT} = & -\varepsilon_k'' I_{\mathbf{k}} + \text{Im} \sum_{\mathbf{k}_1} \left[\frac{4\varepsilon_{\mathbf{k}_1, \mathbf{k}-\mathbf{k}_1}^{(2)} \varepsilon_{-\mathbf{k}_1, \mathbf{k}}^{(2)}}{\varepsilon_{\mathbf{k}-\mathbf{k}_1}^1 (\omega_{\mathbf{k}} - \omega_{\mathbf{k}_1})} - 2\varepsilon_{\mathbf{k}_1, -\mathbf{k}_1, \mathbf{k}}^{(3)} \right] I_{\mathbf{k}_1} I_{\mathbf{k}} \\ & + 2 \text{Im} \sum_{\mathbf{k}_1} \frac{|\varepsilon_{\mathbf{k}_1, \mathbf{k}-\mathbf{k}_1}^{(2)}|^2}{\varepsilon_{\mathbf{k}}^{1*}} I_{\mathbf{k}_1} I_{\mathbf{k}-\mathbf{k}_1} \end{aligned} \quad (9)$$

where $\varepsilon_{\mathbf{k}}^1 = \varepsilon_{\mathbf{k}}' + i\varepsilon_{\mathbf{k}}''$ and $\varepsilon^{2,3}$ are evaluated at $\omega = \omega_{\mathbf{k}}$.

The integrals in $\text{Im } \varepsilon^{(3)}$, too complicated to discuss here, are studied extensively in Refs. 21 and 22. The results are that the third-order contribution is comparable with the linear contribution,

$$\text{Im} \sum_{\mathbf{k}_1} \varepsilon_{\mathbf{k}_1, -\mathbf{k}_1, \mathbf{k}}^{(3)} |\varphi_{\mathbf{k}_1}|^2 \simeq (W/nT_e)(kv_e/\omega_k)^2 \simeq \text{Im } \varepsilon_{\mathbf{k}}^1$$

for

$$\frac{W^{\text{crit}}}{nT_e} \sim \left(\frac{\omega_k}{kv_e} \right)^2 \sim \frac{m_e}{m_i} \quad (10)$$

For $W > W^{\text{crit}}$ the third-order contribution from resonant electrons dominates in Eqs. (7) and (9). We conclude that weak turbulence theory is not valid for ion acoustic turbulence with $W > W^{\text{crit}}$ because of the third-order truncation in Eq. (6).

Problems in addition to the electron divergence discussed above occur when applying weak turbulence theory to ion acoustic turbulence. The principal problems in using weak turbulence theory are summarized briefly as follows:

(1) The electron-wave interaction series is divergent for $W/nT_e > m_e/m_i$.

(2) The weak turbulence wave spectrum $I(\mathbf{k})$ determined by induced wave scattering requires a long-wavelength cutoff⁽²³⁾ for finite $W/nT_e = \int d\mathbf{k} (1 + \mathbf{k}^2 \lambda_{De}^2) I(\mathbf{k})$. A long-wavelength cutoff is observed in collisionless experiments^(6,9,10) and simulations.^(13,14)

(3) Weak turbulence theory predicts no significant production of fast ions during a turbulent heating pulse. (Once the fast ion component is established quasilinear theory adequately describes its evolution).

(4) Weak turbulence theory fails to describe the finite lifetime of the waves. A related problem is the absence of a three-wave resonance contribution to the theory since $\delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}_1} - \omega_{\mathbf{k} - \mathbf{k}_1}) = 0$ for the linear $\omega(\mathbf{k})$.

4. RENORMALIZED TURBULENCE THEORY

To eliminate the divergence of the small-amplitude expansion for $\delta f_k^{(n)}(\mathbf{v})$ it is necessary to select the dominant secular terms in the $\omega = \mathbf{k} \cdot \mathbf{v}$ resonance at n th order in the expansion and sum their contribution to all orders. The n th-order contribution is

$$f_k^{(n)}(\mathbf{v}) = \underbrace{\hspace{1.5cm}}_k \downarrow + \underbrace{\hspace{2.5cm}}_k \begin{array}{c} |k_1 \\ \downarrow \\ k - k_1 \end{array} + \dots + \underbrace{\hspace{2.5cm}}_k \begin{array}{c} |k_1 \\ \downarrow \\ k - k_1 \end{array} \underbrace{\hspace{1.5cm}}_{k_2 \dots} \downarrow k_n \quad (11)$$

and the highest-order multiple resonance in g_k^0 is given by

$$\delta f_k^{(n)}(\mathbf{v}) = \text{---} \downarrow^k \text{---} + \text{---} \begin{array}{c} k_1 \text{---} \square \text{---} -k_1 \\ k \qquad k-k_1 \end{array} \downarrow^k + \text{---} \begin{array}{c} \square \qquad \square \qquad \dots \\ k \qquad k \qquad k \end{array} \downarrow^k \quad (12)$$

+ subdominant terms

$$= -\left(\frac{e}{m}\right) \left[\omega - \mathbf{k} \cdot \mathbf{v} + \left(\frac{e}{m}\right)^2 \frac{\partial}{\partial \mathbf{v}} \cdot \sum_{k_1} I_{k_1} \mathbf{k}_1 \mathbf{k}_1 g_{k-k_1}^0 \frac{\partial}{\partial \mathbf{v}} \right]^{-1} \cdot \frac{\partial \langle f \rangle}{\partial \mathbf{v}}$$

+ subdominant terms

The selection and summation of these terms is given by Choi and Horton^(18,19) and is called the simply renormalized propagator $\tilde{g}_k(\mathbf{v})$. The approximation is also studied by Misguich and Balescu⁽²⁴⁾ and is called the weak coupling approximation.

Other higher-order but less secular terms can also be included in the summation (11) for the renormalized propagator. In the next approximation, called the doubly renormalized propagator $\tilde{\tilde{g}}_k(\mathbf{v})$, the additional fourth-order term $\square \square$ is added to $\square \square$. Retaining the largest class of terms that can be formally summed in the propagator series leads to the nonlinear operator equation^(18,25)

$$\left[\omega - \mathbf{k} \cdot \mathbf{v} + \left(\frac{e}{m}\right)^2 \frac{\partial}{\partial \mathbf{v}} \cdot \sum_{k_1} I_{k_1} \mathbf{k}_1 \mathbf{k}_1 g_{k-k_1}(\mathbf{v}) \cdot \frac{\partial}{\partial \mathbf{v}} \right] g_k(\mathbf{v}) = I \quad (13)$$

for the fully renormalized propagator $g_k(\mathbf{v}) = (\omega - \mathbf{k} \cdot \mathbf{v} + i\nu_k)^{-1}$. The simply renormalized propagator (12) can be calculated exactly in terms of the spectrum $I_{\mathbf{k}\omega}$, whereas the nonlinear operator equation (13) for the fully renormalized propagator is not solvable.

In addition to the propagator series selected in Eq. (12) which has the interaction $g_k^0 \mathbf{E}_k \cdot \partial_{\mathbf{v}}$ as a terminal line, there are terms in the series where the contribution $g_k^0 \mathbf{E}_k \cdot \partial_{\mathbf{v}}$ is not terminal. The series formed by these terms leads to the renormalized vertex operator^(20,26,27)

$$\xi_k(\mathbf{v}) = \frac{\partial}{\partial \mathbf{v}} + \left(\frac{e}{m}\right)^2 \sum_{k_1} I_{k_1} \mathbf{k}_1 \cdot \frac{\partial}{\partial \mathbf{v}} g_{k-k_1} \xi_k g_{-k_1} \mathbf{k}_1 \frac{\partial}{\partial \mathbf{v}} \quad (14)$$

The vertex contribution is well known for drift waves.⁽²⁸⁾

With these selective summations of the perturbation series the renormalized fluctuating distribution function is

$$\delta f_k(\mathbf{v}) = -\frac{e}{m} \varphi_k g_k(\mathbf{v}) \mathbf{k} \cdot \xi_k \langle f \rangle \quad (15)$$

In addition to the coherent part $\delta f_k(\mathbf{v})$ of the fluctuation given by (15) there is an incoherent contribution given by Dupree⁽²⁷⁾ arising from non-wave fluctuations called clumps and holes. The role of holes in $d=1$ ion acoustic turbulence is considered in Section 5.

Now we reconsider the convergence of the perturbation expansion of Eq. (11) in powers of $(e/m) g_k \varphi_k \mathbf{k} \cdot \partial_{\mathbf{v}}$ using $g_k = (\omega - \mathbf{k} \cdot \mathbf{v} + iv_k)^{-1}$ given by Eq. (13). The new expansion operator is bounded by

$$\varepsilon = \max \left\| \frac{(e/m)\varphi_k}{\omega - \mathbf{k} \cdot \mathbf{v} + iv_k} \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{v}} \right\| = \left| \frac{ek\varphi_k}{mv_k(v_k/k)} \right| \leq 1 \quad (16)$$

where to obtain the inequality the smallest value of v_k allowed by Eq. (13) is $v_k \geq (e/m)^2 k^2 \varphi_k^2 / v_k \Delta v_k^2$ and $\Delta v_k = v_k/k$ to give $v_k^2 \geq (e/m) k^2 |\varphi_k|$.

The new or renormalized perturbation series appears at least asymptotically convergent for the levels of turbulence required to describe strong ion acoustic turbulence. The renormalized expansion eliminates the divergence problem (1) of Section 3.

For electrons, the turbulent collision operator iv_k in the renormalization series (12) describes angular scattering of the electrons by the ion acoustic fluctuations. Noting that $\mathbf{k} \cdot \mathbf{v} / \omega_k \simeq (m_e/m_i)^{1/2}$ for thermal electrons the impulse $\Delta \mathbf{v} = e\mathbf{k}\varphi_k/m_e v_k$ imparted to a resonant electron is essentially perpendicular to \mathbf{v} . The electron propagator reduces to

$$(g_k^e)^{-1} = \omega - \mathbf{k} \cdot \mathbf{v} - iv_{\text{eff}} \left(\frac{v_e}{v} \right)^3 \nabla_{\Omega_e}^2 \quad (17)$$

with

$$v_{\text{eff}} = \frac{\pi v_e}{4} \int d\mathbf{k}_1 |\mathbf{k}_1| \frac{e^2 I_{\mathbf{k}_1}}{T_e^2} \simeq c_1 \omega_{pe} \left(\frac{W}{nT_e} \right) \quad (18)$$

where the constant c_1 depends on the shape of the spectrum. For turbulence with $W/nT_e \sim (m_e/m_i)^{1/2}$ the effective Lorentz collision frequency from the turbulence is $v_{\text{eff}} \sim \omega_{pi}$ in agreement with strong turbulence experiments.⁽⁶⁻⁹⁾

Analysis of the electron contribution to the renormalized dielectric function⁽³⁰⁾ $\tilde{\varepsilon}_{\mathbf{k}}(\omega, W)$ shows that at long wavelengths $\lambda = 2\pi/k > v_e/v_{\text{eff}}$ the

Landau resonance is disrupted by the turbulent scattering. In the region $k\lambda_{De} \leq v_{\text{eff}}/\omega_{pe} \sim W/nT_e$ the ion Landau damping dominates and the wave energy is absorbed. The absorption leads to a low- k cutoff of the spectrum $I(k)$ and eliminates the long-wavelength divergence problem (2) of Section 3. The details of the $\tilde{\epsilon}_k(\omega, W)$ and $I(k)$ calculations are given in Refs. 18, 19, 30. A typical wave number spectrum is shown in Fig. 2.

The ion interaction with the turbulence is dominated by the high-energy ions since $\omega_k/k \gg v_i = (T_i/m_i)^{1/2}$. The renormalized ion propagator is calculated from velocity diffusion taking $D_{\parallel}(k) \hat{v}_d \hat{v}_d + D_{\perp}(k)(I - \hat{v}_d \hat{v}_d)$. The explicit formulas for $\tilde{g}_{k\omega}^i(\mathbf{v}, \mathbf{v}')$ are given in Section 6 of Ref. 19. The turbulent propagator $g_{k\omega}(\mathbf{v}, \mathbf{v}')$ and the ω transform $g_k(\mathbf{v}, \mathbf{v}', \tau)$ form a Markov semigroup with

$$\int d\mathbf{v}' g_k(\mathbf{v}, \mathbf{v}', \tau_1) g_k(\mathbf{v}', \mathbf{v}'', \tau_2) = g_k(\mathbf{v}, \mathbf{v}'', \tau_1 + \tau_2)$$

for $\tau_1 + \tau_2 > 0$ and zero for τ_1 or $\tau_2 < 0$. From the analysis of the two-point phase space correlation function $f(\mathbf{x}_1 \mathbf{v}_1; \mathbf{x}_2 \mathbf{v}_2, \tau)$ Dupree⁽²⁹⁾ shows that neighboring trajectories in phase space diverge exponentially with the separation increasing as $k |\delta x| + |\delta v/v_i| \sim \exp[(k^2 D_k)^{1/3} t]$. The result is an

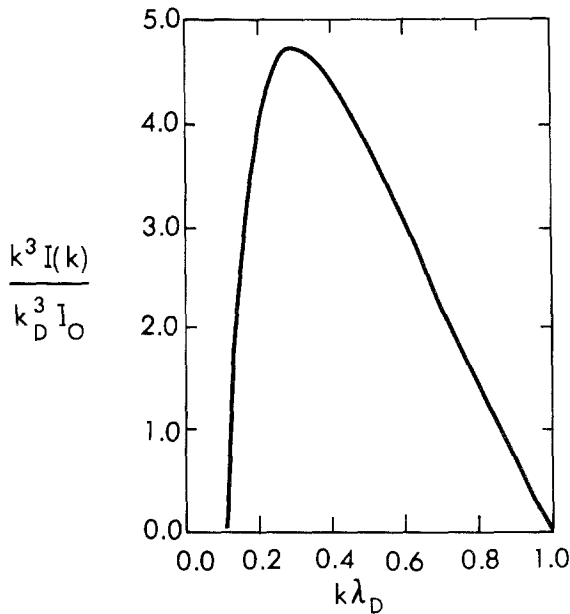


Fig. 2. Wavenumber spectrum from analytic solution of renormalized mode-coupling equation taking square box angular distribution with $\cos \vartheta_0 = c_s/v_d$.

exponential sensitivity to the initial data which gives rise to the statistical Markovian behavior for times greater than the Lyapunov time $t > \tau_c = (k^2 D_k)^{-1/3}$ describing the maximum lifetime of wavelike correlations.

The renormalized propagator $g_k(\mathbf{v}, \mathbf{v}', \tau)$ describes a dissipative dynamics which opens new regions of \mathbf{k} , \mathbf{v} , ω space to wave-particle interactions. Although the net dissipation is constrained by frequency sum rules,⁽³¹⁾ the interactions now extend well outside the region $\omega = \mathbf{k} \cdot \mathbf{v}$. The evolution of the background ion distribution is given by

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \cdot \int d\mathbf{v}' D(\mathbf{v}, \mathbf{v}', \tau) \cdot \frac{\partial}{\partial \mathbf{v}'} \langle f(\mathbf{v}', t) \rangle \quad (19)$$

with the nonlocality of $D(\mathbf{v}, \mathbf{v}')$ being strong for $W/nT \gtrsim 10^{-2}$. Calculations of the ion acceleration from Eq. (19) show that the renormalization eliminates problem (3) of Section 3.

The charge density $\rho_k^{(n)} = \sum e \int \delta f_k^{(n)} d\mathbf{v}$ computed from the renormalized perturbation expansion yields the mode-coupling equation (7) where the formulas for $\varepsilon_{k_1 \dots k_n}^{(n)}$ contain $g_k(\mathbf{v}, W)$. Iterating the equation and neglecting the fourth cumulant (8) also leads to Eq. (9). For long wavelengths, however, the procedure fails owing to the divergence from

$$\varphi_k^{(2)} = - \frac{\varepsilon_{\mathbf{k}_1, \mathbf{k} - \mathbf{k}_1}^{(2)} \varphi_{\mathbf{k}_1} \varphi_{\mathbf{k} - \mathbf{k}_1}}{(\omega_{\mathbf{k}} - \omega_{\mathbf{k}_1} - \omega_{\mathbf{k} - \mathbf{k}_1})(\partial \varepsilon / \partial \omega_{\mathbf{k}})} \rightarrow \infty \quad \text{as } k, k_1 \rightarrow 0 \quad (20)$$

and the higher-order iterations of this fluctuation propagator.

The correlations from the near-three-wave resonance (20) at long wavelengths leads to the formation of intrinsic high-order correlation as contained in solitons. The soliton components of the field are discussed in Section 5.

Assuming the fields remain sufficiently random that a statistical description remains valid we may sum the divergence from $\varepsilon_{k_1 k_2}^{(2)} \varphi_{k_1} \varphi_{k_2} / \varepsilon_k^1$ to all orders to obtain a renormalized fluctuation propagator $\varepsilon_k^{\text{nl}}$. The renormalized dispersion function satisfies the equation

$$\varepsilon_k^{\text{nl}} = \varepsilon_k^1 - \sum_{k_1 = \mathbf{k}_1 \omega_1} \left(\frac{4\varepsilon_{k_1, k - k_1}^{(2)} \varepsilon_{-k_1, k}^{(2)}}{\varepsilon_{k - k_1}^{\text{nl}}} - 2\varepsilon_{k_1, -k_1, k}^{(3)} \right) I_{k_1} \quad (21)$$

and the spectral equation becomes

$$|\varepsilon_k^{\text{nl}}|^2 I_k = 2 \sum_{k_1} |\varepsilon_{k_1, k - k_1}^{(2)}|^2 I_{k_1} I_{k - k_1} \quad (22)$$

Equations (21) and (22) are the equivalent of the direct interaction approximation (DIA) of fluid turbulence with respect to the mode coupling $\varepsilon_{k_1, k_2}^{(2)}$. The equations in this form were first analyzed for ion acoustic turbulence by Tsytovich.⁽³²⁾ Our mode simulation studies indicate that the effects of the three-wave coupling terms are less important, because of the small \mathbf{k} space volume for divergence (20), than the induced wave scattering process in determining the shape of the wave number spectrum $I(\mathbf{k}) = \int_{-\infty}^{+\infty} d\omega/2\pi I(\mathbf{k}, \omega)$ which is a cone in the direction of \mathbf{v}_d .

The near-three-wave resonance is important in determining the frequency linewidth through Eqs. (21) and (22). The linewidth $\Delta\omega_{\mathbf{k}}$ is given approximately by $\Delta\omega_{\mathbf{k}} = -\text{Im} \varepsilon_{\mathbf{k}}^{\text{nl}}/\partial\varepsilon^{\text{nl}}/\partial\omega_{\mathbf{k}}$ and the spectral distribution is approximately

$$I_{\mathbf{k}}(\omega) = \frac{2\Delta\omega_{\mathbf{k}} I_{\mathbf{k}}}{(\omega - \omega_{\mathbf{k}})^2 + \Delta\omega_{\mathbf{k}}^2} \tag{23}$$

The complete solution of Eqs. (21) and (22) for the spectrum $I_{\mathbf{k}\omega}$ and non-linear dielectric $\varepsilon_{\mathbf{k}}^{\text{nl}}(\omega)$ remains a difficult problem. The approximate solution (23) with $\Delta\omega_{\mathbf{k}} \sim \omega_{\mathbf{k}}(W/nT_e)$ eliminates the finite lifetime problem (4) of Section 3.

The normalized turbulence theory given here is the direct renormalization of weak turbulence theory. An advantage of the theory is its ability to directly predict the quantities of interest to experiments and simulations. A number of comparisons have been made such as with Stenzel,⁽⁶⁾ Kawai *et al.*,⁽⁹⁾ and Slusher *et al.*,⁽¹⁰⁾ and the simulations of Biskamp *et al.*^(13,14) The scaling of the anomalous resistivity predicted by the theory is shown in Fig. 3.

There are other statistical theories of Vlasov turbulence principally those which are developed from applying the direct interaction approximation to the Vlasov–Poisson equations. The first attempt with the DIA was given by Orszag and Kraichnan.⁽³³⁾ Subsequently, a systematic theory based on the DIA is given by DuBois and Espedal⁽²⁶⁾ and developed further by DuBois.⁽²⁷⁾ The theory contains additional nonlocal velocity space correlations arising from the shielding clouds of particles contained in the response function $R_{12} = \delta f_{k_1}/\delta\phi_{k_2}$ and the fluctuation propagator $1/\varepsilon_{\mathbf{k}}^{\text{nl}}$. The new contributions can be interpreted in terms of quasiparticles as in many body field theories. An example of these quasiparticle contributions and their relation to the test particle propagator $g_{\mathbf{k}}(\mathbf{v})$ analyzed here is shown diagrammatically in Figs. 1 and 2 of DuBois.⁽²⁷⁾ It remains a difficult problem to evaluate the effects of these nonlocal shielding or polarization contributions contained in the DIA theory of ion acoustic turbulence.

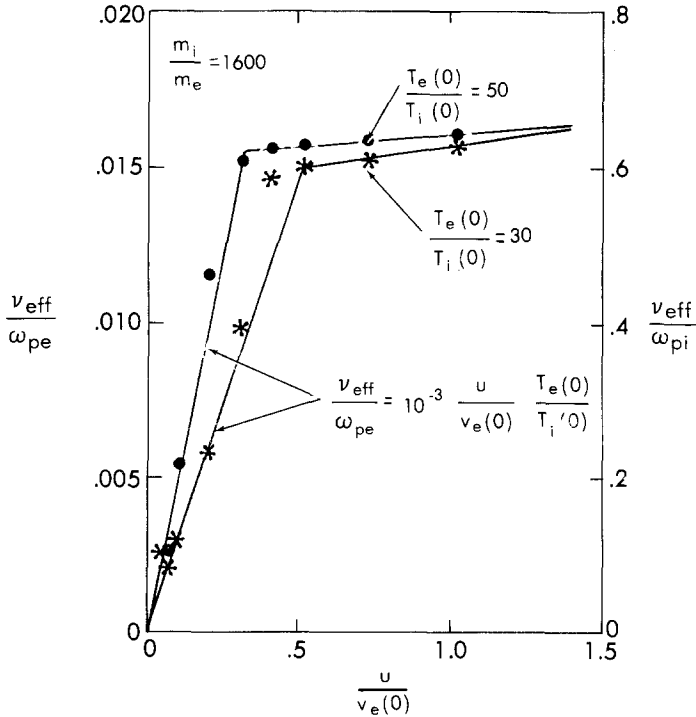


Fig. 3. Anomalous resistivity at maximum of turbulent heating pulse.

5. NONWAVE CONSTITUENTS

Renormalized turbulence theory retains the basic description of weak turbulence theory of a gas of interacting waves and particles. By renormalization of the particles. By renormalization of the particle propagator $g_{\mathbf{k}\omega}(\mathbf{v}, W)$ and the fluctuation propagator $\epsilon_{\mathbf{k}}^{\text{nl}}(\omega, W)$ the secular growth from the bare resonant interactions are eliminated. The renormalization sums to all order the divergent contributions to the perturbation series. The renormalized theory is capable of calculating the quantities of interest in plasma turbulence when high-order correlations are not required. The closure of the hierarchy of correlation functions in renormalized turbulence theory loses the high-order correlations contained in solitons, double layers, and phase space ion holes.

A one-dimensional, long-wavelength ion acoustic wave steepens owing to $v\partial_x v$ until wave dispersion at the Debye scale balances the steepening.

The balance of dispersion and the steepening leads to the spectrum

$$\varphi_s(k, \omega) = \pi k \delta(\omega - kc) \operatorname{csch} \left[\pi k \left(\frac{kc_s}{\omega - kc_s} \right)^{1/2} \right] \exp(-ikx_0) \quad (24)$$

of correlated fluctuation components describing a $\operatorname{sech}^2[K(x - ct)]$ soliton. A similar situation applies to the double layer where the localized potential now contains a net jump $A\varphi$ across the structure.

The ion acoustic solitons preserve their identity for many soliton-soliton and soliton-wave collisions. This coherence property has been used to construct an ideal gas model⁽³⁴⁾ for plasma fluctuations composed of randomly distributed solitons. The soliton component of the fluctuation spectrum for such a gas is given by $I(k, \omega) = k^2 f_s(\omega/k) \operatorname{csch}^2 \{ \pi k [(kc_s/(\omega - kc_s))]^{1/2} \}$ where $f_s(v) dv$ is the number of solitons with the speed $v \geq c_s$ in the range dv . The Gibbs ensemble with $E = H(\varphi, \partial_x \varphi)$ may be used to estimate $f_s(v)$.

Recently studies⁽³⁵⁾ have considered the effect of adding linear dissipative terms modeling the growth and damping $\gamma(k)$ taken from linear particle resonances into the soliton equation. Numerical simulations show that a mixture of solitons and wave components are produced from the unstable growth of noise in this dissipative soliton system.

A more realistic model for coherent structures in ion acoustic turbulence is given by the Kadomtsev-Petviashvili⁽³⁶⁾ equation for two-dimensional solitons. For the cone of waves propagating with the mean angle $\vartheta \ll 1$ with respect to the drift velocity v_d the dispersion in wave frequencies is

$$\omega(\mathbf{k}) = c_s \left(k_z - k_z^3 + \frac{k_\perp^2}{2k_z} \right)$$

Taking into account the wave steepening leads to

$$\frac{\partial^2 \varphi}{\partial z \partial t} + \frac{\partial^2 \varphi}{\partial z^2} + \frac{1}{2} \nabla_\perp^2 \varphi + \frac{\partial^4 \varphi}{\partial z^4} + \frac{\partial}{\partial z} \left(\varphi \frac{\partial \varphi}{\partial z} \right) = \hat{\gamma} \frac{\partial \varphi}{\partial z} F(z, t) \quad (25)$$

with the growth rate $\hat{\gamma}\varphi$ for the unstable dissipative system and $F(t)$ for the forced conservative system.

There are exact two-dimensional soliton solutions of Eq. (25). The equation is used to prove the transverse stability⁽³⁶⁾ of the one-dimensional solitons propagating parallel to $v_d \hat{z}$. The time scale t_s for the formulation of the soliton is given by

$$\frac{1}{t_s} = c_s \langle k_z \rangle \left(\frac{W}{nT_e} \right)^{1/2}$$

and balances the transverse dispersion of the waves for

$$\frac{W}{nT_e} \simeq \left(\frac{\langle k_x^2 \rangle}{\langle k_z^2 \rangle} \right)^2 \simeq \mathfrak{A}^4 \quad (26)$$

The interaction of three obliquely propagating solitons is strong when the ω, \mathbf{k} defined by $\text{sech}^2(\mathbf{k} \cdot \mathbf{x} - \omega t)$ satisfy the three-wave resonance condition.⁽³⁸⁾ When the resonance condition is satisfied two solitons collide to produce a third soliton.

Experimental studies^(39,40) showing the oblique collisions of ion acoustic solitons have been performed confirming the resonance condition and other properties. Clearly, with a large number of obliquely colliding solitons the system may evolve to a chaotic state with strong correlations.

Another important type of correlation that lives for times long compared with the lifetime of the wave constituents is the phase space hole.^(41,42) The role of ion holes has been demonstrated in $d=1$ particle simulations of current-driven ion acoustic turbulence.⁽⁴³⁾ The simulation shows that the production of holes starts for v_d below the threshold for unstable waves from a nonlinear instability.

Making a hole $\delta f_i(x, v) < 0$ in the ion phase space distribution of size $\Delta v \simeq (e\delta\phi/m_i)^{1/2}$ and $\Delta x \sim 10\lambda_{Di}$ leads to a long-lived fluctuation $\delta\phi(x - vt)$ for $v \leq v_i$ with

$$\left(\nabla^2 - \frac{1}{\lambda_D^2} \right) \delta\phi \simeq -\frac{1}{\lambda_D^2} \delta\phi = -4\pi e_i \delta f_i \Delta v \quad (27)$$

The negative potential $\delta\phi$ of the hole reflects electrons $v_e \rightarrow -v_e$ and in the process gains momentum $2m_e \Delta v_e$. Calculating the imbalance of the right- and left-going momentum transfer due to the drift v_d of the electron distribution leads to the hole growth rate

$$\gamma^{\text{hole}} = -8 \left(\frac{\Delta v}{\Delta x} \right) \Delta x^4 \omega_{pi}^2 \omega_{pe}^2 \frac{\partial f_e}{\partial v} \frac{\partial f_i}{\partial v} \simeq \omega_{bi} \left(\frac{\Delta x}{\lambda_D} \right)^4 \left(\frac{v v_d}{v_i v_e} \right) \quad (28)$$

for holes with $0 < v < v_d$.

The hole turbulence appears as fluctuation components with $\omega \sim kv_i$, a region of heavy damping for linear waves. Probably, the most important aspect of the hole turbulence phenomena is the possibility that it relaxes the onset conditions for the occurrence of ion acoustic turbulence and the associated transport processes. There are, in fact, numerous experiments with indications of ion acoustic turbulence where the conditions on T_i/T_e and v_d/c_s for unstable waves are not satisfied.

We conclude that although the phenomena of wave-particle interactions in the context of renormalized turbulence theory has given formulas for calculating \mathbf{k} , \mathbf{v} , ω spectra and transport coefficients, as given in Figs. 1-3, for example, the role of solitons and phase space holes, especially as they interact with the wave fluctuation spectrum, remains to be evaluated.

REFERENCES

1. R. Z. Sagdeev, in *Review of Plasma Physics*, Vol. 4, M. A. Leontovich, ed. (Consultants Bureau, New York, 1966), pp. 23-90.
2. F. V. Coroniti and A. Eviator, *Astrophys. J. Supp.* **33**:189 (1977).
3. J. D. Huba, N. T. Gladd, and K. Papadopoulos, *J. Geophys. Res.* **83**:5217 (1978).
4. K. Papadopoulos, in *Dynamics of the Magnetosphere*, S. I. Akasofu, ed. (D. Reidel, Dordrecht, 1979), p. 289.
5. R. L. Stenzel, W. Gekelman, and N. Wild, *J. Geophys. Res.* **87**:111 (1982).
6. R. L. Stenzel, *Phys. Fluids* **21**:91 (1978).
7. R. D. Bengtson, K. W. Gentle, J. Jancarik, S. S. Medley, P. Nielsen, and P. Phillips, *Phys. Fluids* **18**:710 (1975).
8. S. M. Hamberger and J. Jancarik, *Phys. Fluids* **15**:825 (1972).
9. Y. Kawai, Ch. Hollenstein, and M. Guyot, *Phys. Fluids* **21**:970 (1978).
10. R. E. Slusher, C. M. Surko, D. R. Moler, and M. Porkolab, *Phys. Rev. Lett.* **36**:674 (1976).
11. A. A. Galeev and R. Z. Sagdeev, in *Reviews of Plasma Physics*, Vol. 7, M. A. Leontovich, ed. (Consultants Bureau, New York, 1979), pp. 141-180.
12. G. E. Vekstein, D. D. Ryutov, and R. Z. Sagdeev, *Zh. Eksp. Teor. Fiz.* **60**:2142 (1971) [*Sov. Phys.-JETP* **33**:1152 (1971)].
13. C. T. Dum, R. Chodura, and D. Biskamp, *Phys. Rev. Lett.* **32**:1231 (1974).
14. D. Biskamp, R. Chodura, and C. T. Dum, *Phys. Rev. Lett.* **34**:131 (1975).
15. D. Choi and W. Horton, *Phys. Fluids* **18**:858 (1975).
16. P. L. Mascheroni, *Phys. Fluids* **20**:634 (1977).
17. L. M. Altshul and V. I. Karpman, *Zh. Eksp. Teor. Fiz.* **49**:515 (1965). [*Sov. Phys.-JETP* **22**:361 (1966)].
18. D. Choi and W. Horton, *Phys. Fluids* **17**:2048 (1974).
19. W. Horton and D. I. Choi, *Phys. Rep.* **49**:273-410 (1979).
20. R. Balescu and J. H. Misquich, in *Statistical Physics and Chaos in Fusion Plasmas*, C. W. Horton and L. E. Reichl, eds. (Wiley, New York, 1984) pp. 295-310.
21. V. L. Sizonenko and K. N. Stepanov, *Zh. Eksp. Teor. Fiz. Pis. Red.* **9**:282 (1969) [*J. Exp. Theor. Phys. Lett* **9**:165 (1969)]; *Nucl. Fusion* **19**:155 (1970).
22. D. Choi and W. Horton, *Phys. Fluids* **20**:628 (1977).
23. B. B. Kadomtsev, *Plasma Turbulence* (Academic Press, London, 1965), p. 71.
24. J. H. Misquich and R. Balescu, *J. Plasma Phys.* **19**:147 (1978); *Plasma Phys.* **20**:781 (1978).
25. L. I. Rudakov and V. N. Tsytovich, *Plasma Phys.* **13**:213 (1971).
26. D. F. DuBois and M. Espedal, *Plasma Phys.* **20**:1209 (1978).
27. D. F. DuBois, *Phys. Rev. A*, **865** (1981).
28. T. H. Dupree and D. J. Tetreault, *Phys. Fluids* **21**:425 (1978).
29. T. H. Dupree, *Phys. Fluids* **15**:334 (1972).
30. R. A. Koch and W. Horton, *Phys. Fluids* **18**:861 (1975).

31. C. W. Horton, in *Long-Time Prediction in Dynamics*, C. W. Horton, L. E. Reichl, and V. G. Szebehely, eds. (Wiley, New York, 1983), p. 310–312.
32. V. N. Tsytovich, *Plasma Phys.* **13**:74 (1971).
33. S. A. Orszag and R. H. Kraichnan, *Phys. Fluids* **10**:1720 (1967).
34. J. D. Meiss and W. Horton, Jr., *Phys. Rev. Lett* **48**:1362 (1982).
35. J. D. Meiss, T. Kamimura, and W. Horton, Turbulence in the forced RLW equation (unpublished).
36. B. B. Kadomtsev and V. I. Petviashvili, *Dokl. Akad. Nauk SSSR*, **192**:753 (1970) [*Sov. Phys. Dokl.* **15**:539 (1970)].
37. B. B. Kadomtsev and V. I. Petviashvili, *Sov. Phys. Dokl.* **18**:115 (1973).
38. J. W. Miles, *J. Fluid Mech.* **79**:171 (1979).
39. Y. Nishida and T. Nagasawa, *Phys. Rev. Lett.* **45**:1626 (1980).
40. I. Tsukabayashi and Y. N. Nakamura, *Phys. Lett.* **85A**:152 (1981).
41. T. H. Dupree, *Phys. Fluids* **26**:2460 (1983).
42. D. J. Tetreault, *Phys. Fluids* **26**:3247 (1983).
43. R. H. Berman, D. J. Tetreault, T. H. Dupree, and T. Boutros-Ghali, *Phys. Rev. Lett.* **48**:1249 (1982).